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23 MAY 1911
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PROGRESS REPORT.

1. The Committee feel great pleasure in announcing for the information of the general body, that Mr. G. S. Rama Aiyar, B.A., B.C.E., Executive Engineer, Kurnool, has become a Life Member of the Society.

2. Mr. V. Guruswami Sastri, B.A., L.T., Mathematics Teacher, Taluka High School, Tenali (Guntur District), has been elected a member of the Society at the concessional rate.

3. The Committee regret to have to remove from the list of members, the name of Mr. R. V. Narayana Aiyar, B.C.E., as he is heavily in arrears of his dues and he has never cared to inform the Committee of his intentions.

4. The following books have been purchased for the Library—

   1. *A Treatise on the Geometry of surfaces* by Basset: 8*s*.
   2. *Lecons sur le Calcul des Variations* by Hadamaid: 15*s*.

5. To meet the increasing demand from members for the issue of "School Science and Mathematics" and "Popular Astronomy," the Committee have thought it advisable to order out *three* copies of these journals instead of *one* as hitherto.

6. In accordance with article X of our New Constitution, the audited Balance Sheet for the year 1910 is published below for the information of the general body. The Budget for the current year is also published for information.

POONA,
1st April 1911.

D. D. KAPADIA,
Honorary Joint Secretary.

Balance Sheet for 1910.

Receipts.	Rs.	A.	P.
Opening Balances	1,105	9	4
Subscriptions	1,590	0	0
Subscriptions to Journal	122	0	0
Miscellaneous	67	0	10
Total Rs. ...	2,884	10	2

Expenditure.	Rs.	A.	P.
Books and Journals	985	10	0
Printing Journal	432	13	0
Ordinary working expenses... ...	531	12	5
Special working expenses	64	4	0
Balances	870	2	9
Total Rs. ...	2,884	10	2

C. POLLARD,
Treasurer.

Auditor's Report.

I have examined the Treasurer's books and vouchers, and the monthly statements submitted by the Secretary, Assistant Secretary, and Assistant Librarian, and declare the above account to be correct.

MADRAS,
11th February 1911.

K. R. RAMASWAMI AIYANGAR,
Auditor.

Budget for 1911.

Receipts.	Rs.	A.	P.
Opening Balances	870	2	9
Subscriptions from Members ...	1,650	0	0
Do. to Journal	96	0	0
Donations to Library	30	0	0
Miscellaneous	33	13	3
Total Rs.	2,680	0	0

Expenditure.	Rs.	A.	P.
Books and Journals	1,200	0	0
Priting of Journal	550	0	0
Ordinary working expenses... ...	480	0	0
Special do. do.	100	0	0
Closing balances	350	0	0
Total Rs.	2,680	0	0

MADRAS,
February 1911.

C. POLLARD,
Treasurer.

OFFICE OF THE ... REGISTERED 23 MAY 1911 MADRAS.

On the motion of a gas in two dimensions
(concluded.)

By A. Steichen, S. J., Dr. Phil.

§ 4. The region of double expansion [1].

(a) *Formation of the main equation.*

The equation of continuity :—

$$\frac{\partial(S\xi)}{\partial x}+\frac{\partial(S\eta)}{\partial y}+\frac{\partial S}{\partial t}=0 \quad (1)$$

or

$$\xi\frac{\partial S}{\partial x}+\eta\frac{\partial S}{\partial y}+S\left(\frac{\partial \xi}{\partial x}+\frac{\partial \eta}{\partial y}\right)+\frac{\partial S}{\partial t}=0$$

we combine with Euler's equations :

$$S\frac{d\xi}{dt}=S\left(\xi\frac{\partial \xi}{\partial x}+\eta\frac{\partial \xi}{\partial y}+\frac{\partial \xi}{\partial}\right)=SX-\frac{\partial p}{\partial x}$$

$$S\frac{d\eta}{dt}=S\left(\xi\frac{\partial \eta}{\partial x}+\eta\frac{\partial \eta}{\partial y}+\frac{\partial \eta}{\partial t}\right)=SY-\frac{\partial p}{\partial y}$$

According to restriction (*f*), $\frac{\partial S}{\partial t}=\frac{\partial \xi}{\partial t}=\frac{\partial \eta}{\partial t}=0$

According to restriction (a), $\frac{\partial S}{\partial x}=\frac{dS}{dp}\frac{\partial p}{\partial x}$ and $\frac{\partial S}{\partial y}=\frac{dS}{dp}\frac{\partial p}{\partial y}$

Also $\frac{dp}{dS}=c^2$

$$\therefore\frac{\partial S}{\partial x}=\frac{1}{c^2}\frac{\partial p}{\partial x} \text{ and } \frac{\partial S}{\partial y}=\frac{1}{c^2}\frac{\partial p}{\partial y} \quad \dots \quad \dots \quad (a)$$

According to restriction (b), $X=Y=0$

Hence we get from Euler's equations

$$-\frac{\partial p}{\partial x}=S\left(\xi\frac{\partial \xi}{\partial x}+\eta\frac{\partial \xi}{\partial y}\right)$$
$$-\frac{\partial p}{\partial y}=S\left(\xi\frac{\partial \eta}{\partial x}+\eta\frac{\partial \eta}{\partial y}\right) \quad \dots \quad \dots \quad (b)$$

Combining the equations (*a*) and (*b*) it follows :

$$\xi\frac{\partial S}{\partial x}=-\xi\frac{1}{c^2}S\left(\xi\frac{\partial \xi}{\partial x}+\eta\frac{\partial \xi}{\partial y}\right)$$
$$\eta\frac{\partial s}{\partial y}=-\eta\frac{1}{c^2}S\left(\xi\frac{\partial \eta}{\partial x}+\eta\frac{\partial \eta}{\partial y}\right)$$

Substituting these values of $\xi\frac{\partial S}{\partial x}$ and $\eta\frac{\partial S}{\partial y}$ in the equation of continuity, we get

$$\frac{\partial \xi}{\partial x}\left(1-\frac{\xi^2}{c^2}\right)+\frac{\partial \eta}{\partial y}\left(1-\frac{\eta^2}{c^2}\right)-\frac{\xi\eta}{c^2}\left(\frac{\partial \xi}{\partial}+\frac{\partial \eta}{\partial x}\right)=0 \quad \dots \quad \text{I}$$

[1] This forms part of my own dissertation, Gottingen 1909.

Equation I is the required equation.

(*b*) *Potential of velocities.*

According to restriction (*e*), $\frac{\partial \xi}{\partial y}=\frac{\partial \eta}{\partial x}$.

Therefore there exists a function ϕ of such a nature that the components of velocity ξ and η may be obtained from it by differentiating ϕ with respect to x and y.

That is, $\xi=\frac{\partial \phi}{\partial x}$ and $\eta=\frac{\partial \phi}{\partial y}$.

Function ϕ we call the *potential* function of velocities.

(*c*) *Trnsformations of equation I.*

Equation I is not linear. It will become linear when we transform it by Legendre's transformation.

(α) Legendre's transformation.

We put

$$\phi=\xi x+\eta y-X(\xi,\eta);$$

$$\therefore\ \xi=\frac{\partial \phi}{\partial x}=\phi_x,$$

$$\eta=\frac{\partial \phi}{\partial y}=\phi_y;$$

$$\frac{\partial \xi}{\partial x}=\phi_{xx}=\frac{X_{\eta\eta}}{X_{\xi\xi}X_{\eta\eta}-X^2_{\xi\eta}}$$

$$=\frac{X_{\eta\eta}}{\Delta},$$

$$\frac{\partial \eta}{\partial y}=\phi_{yy}=\frac{X_{\xi\xi}}{\Delta},$$

$$\frac{\partial \xi}{\partial y}=\frac{\partial \eta}{\partial x}=\phi_{xy}=-\frac{X_{\xi\eta}}{\Delta}.$$

$$X(\xi,\eta)=\xi x+\eta y-\phi(x,y)$$

$$x=\frac{\partial X}{\partial \xi}=X_\xi,$$

$$y=\frac{\partial X}{\partial \eta}=X_\eta;$$

$$X_{\xi\xi}=\frac{\phi_{yy}}{\phi_{xx}\phi_{yy}-\phi^2_{xy}}=\frac{\frac{\partial \eta}{\partial y}}{D},$$

$$X_{\eta\eta}=\frac{\frac{\partial \xi}{\partial x}}{D},$$

$$X_{\xi\eta}=-\frac{\frac{\partial \xi}{\partial y}}{D}=-\frac{\frac{\partial \eta}{\partial x}}{D}.$$

Making these substitutions, equation I takes the form

$$\frac{\partial^2 X}{\partial \xi^2}\left(1-\frac{\eta^2}{c^2}\right)+\frac{\partial^2 X}{\partial \eta^2}\left(1-\frac{\xi^2}{c^2}\right)+2\frac{\xi\eta}{c^2}\frac{\partial^2 X}{\partial \xi\partial \eta}=0 \quad \dots \quad \text{II}$$

Equation II is a linear equation.

(β) Transformation to polar co-ordinates.

Next we transform equation II to polar co-ordinates. Call the radius vector w and the angle between w and the ξ-axis θ. Then

$$\xi=w\cos\theta,\quad \eta=w\sin\theta,\ \text{and}\ w^2=\xi^2+\eta^2$$

Equation II then takes the form

$$\frac{\partial^2 X}{\partial w^2}+\frac{\partial^2 X}{\partial \theta^2}\left(\frac{1}{w^2}-\frac{1}{c^2}\right)+\frac{\partial X}{\partial w}\left(\frac{w}{w^2}-\frac{w}{c^2}\right)=0 \qquad \ldots \quad \text{III}$$

(γ) Transformation to curvilinear coordinates.

Finally we introduce curvilinear coordinates $\lambda=$const. and $\mu=$ const. Suppose λ and μ to be connected with w and θ by the following relations:

$$\lambda=\zeta(w)+\theta \text{ and } \mu=\zeta(w)-\theta\,;$$
$$\therefore \lambda+\mu=2\zeta(w) \text{ and } \lambda-\mu=2\theta.$$

$\zeta(w)$ is an undetermined function. Equation III then takes the form

$$\left(\frac{\partial^2 X}{\partial \mu^2}-\frac{\partial^2 X}{\partial \lambda^2}\right)\left[\left(\frac{\partial \zeta}{\partial w}\right)^2-\frac{w^2-c^2}{w^2c^2}\right]+2\left(\frac{\partial \zeta}{\partial w}\right)^2\frac{\partial^2 X}{\partial\lambda\partial\mu}$$
$$+\left(\frac{\partial X}{\partial \lambda}+\frac{\partial X}{\partial \mu}\right)\left[\frac{\partial^2\zeta}{\partial w^2}+\frac{\partial \zeta}{\partial w}\left(\frac{w}{w^2}-\frac{w}{c^2}\right)\right]=0$$

$\zeta(w)$ is an undetermined function still at our disposal. We determine $\zeta(w)$ so that the coefficient of $\frac{\partial^2 X}{\partial \mu^2}-\frac{\partial^2 X}{\partial \lambda^2}$ becomes zero.

Hence, we put $\frac{\partial \zeta}{\partial w}=\pm\frac{\sqrt{w^2-c^2}}{wc}$

$$\therefore \zeta(w)=\pm\int\frac{\sqrt{w^2-c^2}}{wc}dw+\text{constant.}$$

In order to find this integral, we substitute c' for c. Between c and c' there exists the relation :—

$$c^2=\tfrac{1}{2}(k+1)\left[c'^2-\frac{k-1}{k+1}w^2\right]$$

Substituting these values for c we get

$$\zeta(w)=\pm\int\frac{1}{w}\sqrt{\frac{w^2-c'^2}{c'^2-w^2(k-1)/(k+1)}}\,dw+\text{constant.}$$

Also

$$\frac{\partial\zeta}{\partial w}=\pm\sqrt{\frac{w^2-c^2}{w^2c^2}}\,;$$
$$\therefore \frac{\partial^2\zeta}{\partial w^2}=\pm\frac{c^3-w^3dc/dw}{w^2c^2\sqrt{w^2-c^2}}.$$

dc/dw can be expressed in terms of w in the following way :

$$c^2=\frac{dp}{dS} \text{ and } p=aS^k\,;$$
$$\therefore \frac{dp}{dS}=a\,k\,S^{k-1}$$

From the equation of energy we have

$$\int \frac{dp}{S} + \frac{w^2}{2} = C$$

$$\therefore \int akS^{k-2}\, dS = C - \tfrac{1}{2}w^2,$$

$$\frac{ak}{k-1} S^{k-1} = C - \tfrac{1}{2}w^2 = \frac{c^2}{k-1},$$

$$2\, c\frac{dc}{dw} = -(k-1)w$$

$$\text{or} \quad \frac{dc}{dw} = -\frac{k-1}{2} \cdot \frac{w}{c}.$$

Hence we get

$$\frac{\partial^2 \zeta}{\partial w^2} = \pm \frac{c^4 + \frac{1}{2}(k-1)w^4}{w^2 c^3 \sqrt{w^2 - c^2}}.$$

Substituting these values of $\frac{\partial \zeta}{\partial w}$ and $\frac{\partial^2 \zeta}{\partial w^2}$ in equation III, we get

$$\frac{\partial^2 X}{\partial \lambda \partial \mu} = \mp K \left(\frac{\partial X}{\partial \lambda} + \frac{\partial X}{\partial \mu} \right) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \text{IV}$$

where

$$K = \frac{c^4 + \frac{1}{2}(k-1)w^4 - (w^2 - c^2)^2}{2c(w^2 - c^2)^{\frac{3}{2}}},$$

w and c are functions of p/p_0

$$w^2 = \frac{2k}{k-1} \frac{p_0}{S_0} \left[1 - (p/p_0)^{\frac{k-1}{k}} \right],$$

$$c^2 = k\frac{p_0}{S_0} (p/p_0)^{\frac{k-1}{k}}$$

and $\quad 0 \leqq p/p_0 \leqq 0{\cdot}527, \qquad$ for $w > c$

(δ) The characteristics of equation II.

Monge calls a characteristic of a partial differential equation of the second order a curve in the space of x, y, u along which two consecutive integral-surfaces of the differential equation cut and touch. In general, the projection of this curve upon the xy-plane depends on the choice of the integral u. But in the case of an equation of the type

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} = F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$$

where A, B and C are functions of x and y, the projection of the characteristic upon the xy-plane is independent of the choice of the integral u and already fully determined by the differential equation itself. For the sake of shortness we call the projected curves characteristics. These characteristics are given by the equation

$$C\,dx^2 - 2\,B\,dxdy + A\,dy^2 = O \quad (1)$$

Applying this to equation II we get as equation of the characteristics

$$\left(\frac{d\eta}{d\xi}\right)^2\left(1-\frac{\eta^2}{c^2}\right) - 2\frac{\xi\eta}{c^2}\frac{d\eta}{d\xi} + \left(1-\frac{\xi^2}{c^2}\right) = 0.$$

If instead of c we introduce c' this equation takes the form

$$\left(\frac{d\eta}{d\xi}\right)^2\left\{\frac{k+1}{2}c'^2 - \frac{k-1}{2}\xi^2 - \frac{k+1}{2}\eta^2\right\} - 2\xi\eta\frac{d\eta}{d\xi}$$

$$+\left\{\frac{k+1}{2}c'^2 - \frac{k+1}{2}\xi^2 - \frac{k-1}{2}\eta^2\right\} = O.$$

This equation we transform to polar coordinates by putting

$$\xi = w\cos\Theta \text{ and } \eta = w\sin\Theta$$

Then it takes the form

$$\frac{dw}{d\Theta} = \pm w\frac{\sqrt{c'^2 - w^2(k-1)\,(k+1)}}{w^2 - c'^2}.$$

$$\therefore\ \Theta = \pm\int\frac{1}{w}\sqrt{\frac{w^2-c'^2}{c'^2 - w^2(k-1)\,(k+1)}}\,dw + \text{constant.}$$

It will be noticed that we get the same integral as for our curvi linear co-ordinates. Our curvilinear co-ordinates are the characteristics of the differential equation.

Discussion of the equation of the characteristics:

If we denote by $w_{max.}$ the maximum value of w *i.e.* the greatest possible value of the velocity of the gas, we get the relation

$$w^2_{max.} = \frac{k+1}{k-1}c'^2$$

(*aa*) $w = w_{max}$;

$$\therefore\ \frac{dw}{d\Theta} = O, \text{ and } w = \text{constant}$$

$\therefore$ for $w = w_{max.}$ the characteristic of the differential equation is a circle of radius $w = w_{max.}$ with the origin of the co-ordinates as centre.

(*bb*) $w = c'$

$$\therefore\ \frac{dw}{d\Theta} = \infty \qquad \therefore\ \frac{d\Theta}{dw} = O, \text{ cnd } \Theta = \text{constant.}$$

Therefore the characteristics have cusps on the circle $w = c'$. The tangents at these cusps go through theall origin. For $w = c'$ we get only one family of charateristics. In this special case the differential equation is therefore of the parabolic type.

(1) See Enzykl. der math Wiss. II 1 p. 508.

(*cc*) $c' < w < w_{max}$.

In this case we get two families of real charateristics, the equation is therefore in this special case of the hyperbolic type.

(*dd*) $c > w$.

In this case we get two families of imaginary characteristics, the equation is now of the elliptic type.

As equation of the characteristics we found

$$\Theta = \pm \int \frac{1}{w} \sqrt{\frac{w^2 - c'^2}{c'^2 - w^2(k-1)/(k+1)}}\, dw + \text{constant}.$$

From this equation we get after a number of transformations.

$$2\,\Theta = \pm \left\{ \sqrt{\frac{k+1}{k-1}} \sin^{-1}\left[(k-1)\frac{w^2}{c'^2} - k \right] + \sin^{-1}\left[(k+1)\frac{c'^2}{w^2} - k \right] \right\} + \text{constant}.$$

This equation gives us Θ as a function of w/c'; fig. 4 represents one pair of characteristics.

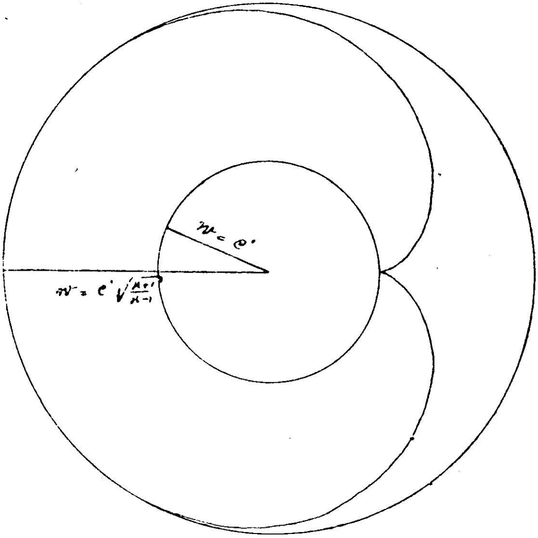


Fig. 4.

The transformation of the differential equation for the characteristics as co-ordinates is of importance, because the characteristics are also the curves along which any disturbance in the gas is propagated.

θ, as we have found, is a function of w/c'. This will enable us also to find θ as a function of ϕ and of p/p_0.

$$w^2=\frac{k+1}{k-1}c'^2\left[1-(p/p_0)^{\frac{k-1}{k}}\right], \dots\dots\dots\dots\dots\dots (a)$$

$$\text{and } \left(\frac{p}{p_0}\right)^{\frac{k-1}{k}}=\frac{2}{k+1}\cos^2\left(\phi\sqrt{\frac{k-1}{k+1}}\right),$$

$$\therefore \frac{w^2}{c'^2}=\frac{k+1}{k-1}\left[1-\frac{2}{k+1}\cos^2\left(\phi\sqrt{\frac{k-1}{k+1}}\right)\right].$$

Substituting this value of w^2/c'^2 in the equation of the characteristics we get the relation between θ and ϕ.

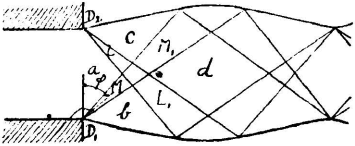
Also from equation (a) we have

$$\frac{w^2}{c'^2}=\frac{k+1}{k-1}\left[1-(p/p_0)^{\frac{k-1}{k}}\right].$$

Substituting this value of w^2/c'^2 into the equation of the characteristics we get the relation between θ and p/p_0

(Σ) Some conclusions.

Without solving the differential equation we arrive at the following conclusions as to the motion of the gas. From the photograph of the jet we get the following diagram in the xy-plane of the moving gas.



FIG, 5.

From the two corners D_1 and D_2 two regions of expansion originate *viz.*, LL_1 and MM_1. They reach the boundaries of the jet and are here

reflected as regions of compression, reach again the boundaries of the jet and are reflected as regions of expansion, and so it goes on.*

In the region a the pressure and the velocity of the gas are constant; moreover the velocity of the gas is greater than the velocity of sound. In the two regions b and c the pressure is equal to the pressure of the outer air, hence the pressure and the velocity are constant.[1]

In the regions L and M there is expansion; hence the velocity and the pressure are variable and are functions of the angle ϕ only.

In the ξ η—plane we get the following diagram.

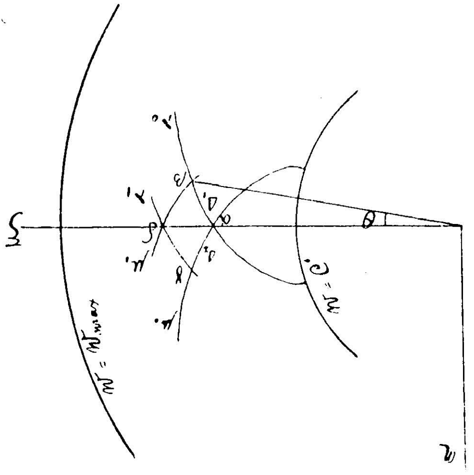


FIG 6.

ξ and η are the components of velocity. Therefore a constant velocity is represented in the ξ η-plane by a single point, and a velocity depending on a single variable only by a curve. Hence corresponding

* Prof. L. Prandtl, Phys. Zeitschr. vol. 8, p. 26.

[1] Dr. Meyer, Diss. p. 10,

to the three constant velocities a, b, c in the xy-plane we have three points α, β, γ in the $\xi\eta$-plane. Corresponding to the two regions of simple expansion L and M in the xy-plane, where the velocity depends on ϕ only, we have in the $\xi\eta$-plane the two curves joining the points α and β, α and γ.

To the whole region d in the xy-plane there corresponds only one point δ in the $\xi\eta$-plane. Hence it follows that in the whole region d, the velocity and the pressure of the gas are constant.

The region bounded by the characteristics passing through the points α, β, γ, δ in the $\xi\eta$-plane corresponds to the region of double expansion in the xy-plane. As we have already pointed out disturbances in the gas are propagated along the characteristics.

The characteristics in the $\xi\eta$-plane are given by our function $\zeta(w)$. At the points α, β, γ and on the curves $\alpha\beta$, $\alpha\gamma$ the velocity is known, hence also $\zeta(w)$ is known at these three points. Let the values of $\zeta(w)$ at these points be ζ_α, ζ_β, ζ_γ.

But we know thst $2\,\zeta_\delta=\lambda_\delta+\mu_\delta$.

The points δ and γ are on the same curve $\lambda=$constant.

$$\therefore\ \lambda_\delta=\lambda_\gamma.$$

Also δ and β are on the same curve $\mu=$constant.

$$\therefore\ \mu_\delta=\mu_\beta\,;$$

$$\therefore\ 2\zeta_\delta=\lambda_\gamma+\mu_\beta=\zeta_\gamma+\theta_\gamma+\zeta_\beta-\theta_\beta\,.$$

Also α and γ are on the same curve $\mu=$constant.

$$\therefore\ \mu_\gamma=\mu_\alpha$$

$$\therefore\ \zeta_\gamma-\theta_\gamma=\zeta_\alpha-\theta_\alpha=\zeta_\alpha,\quad \because\theta_\alpha=0\,;$$

$$\therefore\ \theta_\gamma=\zeta_\gamma-\zeta_\alpha.$$

Similarly $\theta_\beta=\zeta_\alpha-\zeta_\beta\,;$

$$\therefore\zeta_\delta=\zeta_\beta+\zeta_\gamma-\ _\alpha.$$

Hence $\zeta(w)$ is known at the point δ, and consequently also the velocity, the direction of the motion and the pressure of the gas. Since the point δ corresponds to the whole region d in the xy—plane it follows that w and p in this region are known. On account of the assumed symmetry of the jet, p and w are the same at β and γ. Hence from the above equation for ζ_δ, it follows also that w at δ is greater and p is less than at β and γ. Consequently, in the whole region d in the xy—plane the pressure of the gas is constant and less than the pressure of the outer air. If $\zeta_\alpha, \zeta_\beta, \zeta_\gamma$, are known, we can find the value of p in the whole region d.

Moreover, since $2\zeta_i = \lambda_{ik} + \mu_k$, the function $\zeta(w)$ enables us to find for any point (λ_i, μ_k) in the region of double expansion $\alpha\beta\gamma\delta$ the value of $\left(\frac{p}{p_0}\right)_{\lambda_i \mu_k}$

If we assume that in the regions L_1 and M_1, just as in L and M p and w are functions of one variable only, *viz*: of a certain angle, then it follows that L_1 and M_1 correspond to the two curves μ_1 and λ_1 in the $\xi\eta$—plane. There is nothing to prevent us from making this assumption, since L_1 and M_1 are regions of simple expansion. On the curves λ_1 and μ_1, $\zeta(w)$ is known. Hence, without solving the differential equation it is possible to get a qualitative idea of the change of w and p in L_1 and M_1. But the idea is only a qualitative one, since without solving the differential equation we known only the radii-vectores corresponding to the points α, β, γ.

(ζ) Values of $X(\lambda, \mu)$ on the boundary.

In the xy-plane we know p and w at every point of the regions of simple expansion L and M up to the region of double expansion. The two curves Δ_1 and Δ_2 in the $\xi\eta$-plane correspond to these two regions L and M in the xy-plane. On account of Legendre's Transformation, to every point (r, ϕ) in the xy-plane there corresponds a surface-element over the $\xi\eta$-plane and to any radius r in the region of simple expansion L or M a whole system of surface-elements touching the curves Δ_1 and Δ_2, to any point of a given radius r a surface-element of a certain inclination. But in the two regions of simple expansion L and M all the radii pass through a fixed point D_1 or D_2. Hence along the curves Δ_1 and Δ_2 there are surface-elements having all the same inclinations. Therefore the curves Δ_1 and Δ_2 are plane curves. The inclina-

tions of the planes E_1 and E_2 which contain Δ_1 and Δ_2 are given by Legendre's Tranformation

$$\frac{\partial X}{\partial \xi} = x, \text{ and } \frac{\partial X}{\partial \eta} = y.$$

The inclination of the plane E_1 containing the curve Δ_1 is consequently given by the co-ordinates of the point D_1

$$x = o, \qquad y = -d\ ;$$

where $2d$ = distance $D_1\ D_2$

$\therefore$ along the boundary Δ_1 we have

$$\frac{\partial X}{\partial \xi} = o \qquad \frac{\partial X}{\partial \eta} = -d,$$

$$\therefore \qquad X = -\eta d + \text{constant}.$$

Similarly along Δ_2, $X = +\eta d + \text{constant}$.

If we denote by w the vector of velocity and by θ^1 the angle between w and the ξ-axis then we get

$$\eta = w \sin \theta^1.$$

The two curves Δ_1 and Δ_2 are known as functions of w^2/c'^2 and to every point on Δ_1 and Δ_2 there corresponds a certain angle θ connected with θ^1 by the equation

$$\theta' = \theta - \theta_o$$

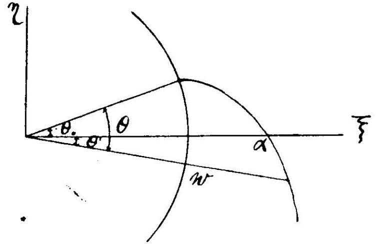


FIG. 7.

Since ζ_a is known, also θ_o is known, hence also θ^1. Therefore it is possible to find η and consequently also X at any point on Δ_1 and Δ_2.

(η) The characteristics and the stream-lines in the xy-plane.

Our chief object is to find the characteristics and the stream-lines in the xy-plane The characteristics in ξ η-plane are known, and the xy-plane is connected with the ξ η-plane by the equations

$$x = \frac{\partial X}{\partial \xi}, \quad y = \frac{\partial X}{\partial \eta};$$

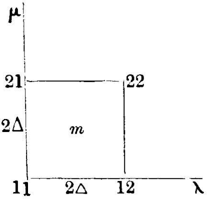
∴ if along a characteristic in the ξ η-plane we form for a number of points $\frac{\partial X}{\partial \xi}$ and $\frac{\partial X}{\partial \eta}$, we get the corresponding points x y of the characteristic in the xy - plane. If, in this way, we draw a net of characteristics in the xy-plane, we get immediately also the stream-lines, since these lines coincide with the lines bisecting the angles between intersecting characteristics.

In order to find $\frac{\partial X}{\partial \xi}$, $\frac{\partial X}{\partial \eta}$ we must first solve the differential equation (IV)

$$\frac{\partial^2 X}{\partial \lambda \, \partial \mu} = \mp K \left\{ \frac{\partial X}{\partial \lambda} + \frac{\partial X}{\partial \mu} \right\}$$ see page 56

§ 5. Numerical integration of the differential equation IV.

According to a method devised by Prof. C. Runge, we substitute quotients of finite differences for the differential coefficients. If we call



2 Δ the finite increment of λ and μ then as a first approximation we get for the point 11

$$\frac{\partial X}{\partial \lambda} = \frac{X_{12} - X_{11}}{2 \Delta}, \qquad \frac{\partial X}{\partial \mu} = \frac{X_{21} - X_{11}}{2 \Delta},$$

$$\frac{\partial^2 X}{\partial \lambda \, \partial \mu} = \frac{(X_{22} - X_{21}) - (X_{12} - X_{11})}{4 \Delta^2},$$

A second and better approximation we get if we form the differential co efficient for the middle point of the square. Here the differential co-efficients are approximately

$$\frac{\partial X}{\partial \lambda}=\frac{(X_{12}-X_{11})+(X_{22}-X_{21})}{4\,\triangle},$$

$$\frac{\partial X}{\partial \mu}=\frac{(X_{21}-X_{11})+(X_{22}-X_{12})}{4\,\triangle},$$

$$\frac{\partial^2 X}{\partial\lambda\,\partial\mu}=\frac{(X_{22}-X_{21})-(X_{12}-X_{11})}{4\,\triangle^2}.$$

The co-efficient K is to be taken at the point m. We denote this by K_m. The differential equation IV then takes the form

$$X_{22}=\frac{X_{21}+X_{11}}{1+2\,\triangle\,K_m}-X_{11}\frac{1-2\,\triangle\,K_m}{1+2\,\triangle\,K_m}\text{............V.}$$

X_{22} is the unknown value of X; X_{11}, X_{12}, X_{21} and K_m are known.

Since the values of X on the two boundaries $\triangle_1$, $\triangle_2$ are known we can by this method find the value of X for as many points in the region $\alpha\beta\gamma\delta$ as we like.

If we expand X_{11}, X_{12}, X_{21}, X_{22} from the point m by Taylor's Theorem and substitute these values in our equation of finite differences, then it will be seen that equation V differs from equation IV only by terms multiplied by $\triangle^2$. Hence, by taking $\triangle$ very small, the error introduced will also be very small.

After having integrated V for the whole region $\alpha\beta\gamma\delta$ it will be easy to find $x=\frac{\partial X}{\partial \xi}$, $y=\frac{\partial X}{\partial \eta}$ by the methods of descriptive geometry.

Example ; $p_0 = 10\ kg/cm^2$

$$\phi_1 = 74^0\,54' \quad \psi_1 = 55^\circ\,23' \quad V_1 = 19^\circ\,31'$$
$$\phi_2 = 89^\circ\,48' \quad \psi_2 = 61^\circ\,18' \quad V_2 = 28^\circ\,31'$$

find the characteristics and the stream-lines in the xy—plane.

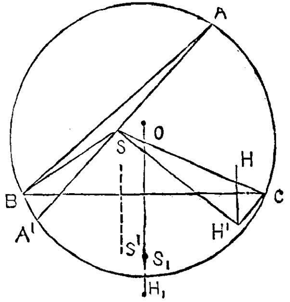
Note.—The above *data* correspond to the jet whose photograph has been given at the beginning of this paper.

The foci of an inconic.

By M. T. Naraniengar.

In this article S, H denote the foci of an in-conic of the triangle ABC. AS, BS, CS, produced meet the circumcircle at A^1, B^1, C^1. S_1, H_1 are the centres of the circles BSC, BHC ; S', H' are the reflections of S and H with respect to BC.

1. *To prove that* S_1, H_1 *are inverse points with respect to the circle ABC.*



Since S_1 is the centre of BSC,

$$BS_1O = \pi - BSC.$$

Similarly, $BH_1O = \pi - BHC$.

$$\therefore\ BS_1O + BH_1O = 2\pi - (BSC + BHC)$$

But $BSC + BHC = \pi + A$ (since S, H are isogonal conjugates)

$$\therefore\ BS_1O + BH_1O = \pi - A = OBH_1 + BH_1O, \text{ as } BOH_1 = A$$

$$\therefore\ BS_1O = OBH_1$$

Hence $OS_1 \cdot OH_1 = OB^2$ and S_1, H_1 are inverse points.

2. *To prove that* BSA^1, H^1SC *are similar.*

If P, Q, R (not shown in the figure) be the points of contact of the conic, then P is evidently the intersection of SH' and BC.

Now, from the focal property of the conic,
$PSC = QSC =$ supplement of $ASB = BSA'$.
Also, $SCH' = SCB + HCB = SCB + SCA = C = SA'B$.
Thus BSA', $H'SC$ are equiangular and similar,
and $SH' . SA' = SB \cdot SC$.

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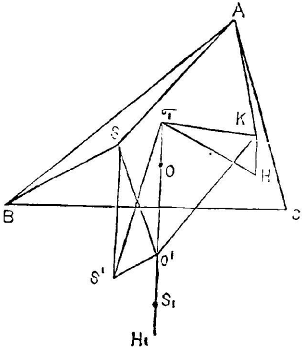
3. *To prove that A′SS′, S_1SH′ are similar.*

Since SS_1, SS′ are equally inclined to SC, SB (a well known property) and $SS_1.SS' = SB.SC$, the triangles BSS′, S_1 SC are similar; and therefore (from § 2), A′SS′, S_1SH′ are also similar.

Cor. A′SS_1, S′SH′ are also similar.

4. Let O^1, σ_1 be the reflections of O, S_1 in BC; complete the parallelogram ASO^1K; *then σ_1 HK is similar to $\sigma_1 S^1 O^1$.*



For, it has been proved that $SS^1.SS_1 = SH^1.SA^1$.

But BC being the radical axis of the circles ABC and BSC, the rectangle $SS^1.OS_1$ is equal to the constant rectangle corresponding to S in the circle ABC; that is,

$$SS^1.\ OS_1 = AS{\cdot}SA^1\ ;$$

$$\therefore\ SS_1/OS_1 = SH^1/SA = S^1H/SA.$$

Thus in the triangles $\sigma_1 S^1 H$, $\sigma_1 O'K$

$$\sigma_1 S^1 : \sigma_1 O^1 = SS_1 : OS_1 = S^1H : SA = S^1H : O^1K.$$

Also, the angle $S^1\sigma_1 O' = S^1SS_1$ = difference between SBC, SCB
= difference between the angles made by SA^1 and SH^1 or S^1H with BC.
= the angle between SA and S^1H
= the angle between O^1K and $S'H$.

Hence, the triangles $\sigma_1 S^1 H$ and $\sigma_1 O^1 K$ are similar.

Therefore also $\sigma_1 S^1 O'$, σ_1HK are similar.

5. *To prove that KH = twice the distance of the N.P. centre from the centre of the in-conic.*

This follows readily, as the N. P. centre is the mid-point of O^1A or SK and the centre of the conic is the mid-point of SH.

6. *To prove that O, σ_1 are inverse points with respect to the circle BHC.*

Since S_1, H_1 are inverses with respect to ABC, $B\sigma_1O = BS_1O = OBH_1$.

Hence, the triangles $B\sigma_1H_1$ and OBH_1 are similar.

$$\therefore\ H_1O.H_1\sigma_1 = H_1B^2.$$

7. *To prove that OS.OH = 2R.d, where d = distance of the N.P. centre from the centre of the in-conic.*

Since H is a point on the circle BHC with respect to which O and σ_1 are inverse points,

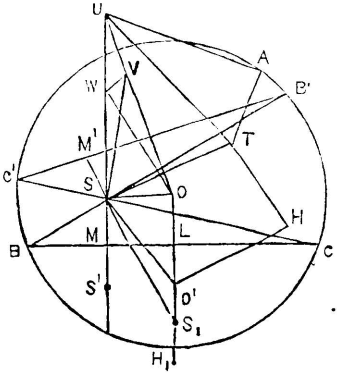
$$OH : \sigma_1H = OH_1 : H_1H = OH_1 : H_1B = OB : BS_1 = R : SS_1$$

But, $\sigma_1H : KH = \sigma_1S' : S'O^1 = SS_1 : SO$,

$$\therefore\ OH : KH = R : SO$$

Hence $$OS.OH = R.KH = 2Rd.$$

8. *To prove that SS_1 is perpendicular to $C'B^1$.*



For, S^1SS_1 = difference between SBC and SCB
= difference between SBC and SB^1C^1
= the angle between BC and C^1B^1

∴ the angle between S_1S and C^1B^1 is a right angle, being the supplement of M.

9. *The intersection of MS and the diameter perpendicular to C^1B^1 is the circumcentre of C^1SB^1.*

This follows from the fact that C^1SB^1 is similar to BSC and S_1 is the centre of BSC. Hence U is the circumcentre of C^1SB^1.

10. *The inverse of U corresponds to O in the directly similar figures C^1AB^1 and BHC*

Let V be this inverse. Then OU. OV = OS_1 . OH,

∴ OV : H_1O = OS_1 : OU
= SU : SS_1, since $OUSS_1$ is a parallelogram.
= C^1B^1 : BC, the triangles C^1SB^1 and BSC being similar.

Hence, OV and H_1O are corresponding segments in the similar figures ; whence the result.

11. *The triangles SUV and SS_1O^1 are similar,*

For UV and S_1O^1 are corresponding segments in the similar figures C^1SB^1 and BSC (by § 10)

12. *If the parallelogram SO^1HT be completed, the triangles UOS and UTA are similar.*

Draw OW parallel to SO^1, then SWO = SO^1O = SVO (by § 11) ;
∴ SWVO is cyclic.

Now, from the corresponding figures C^1AB^1 and BHC
VA : OH or WT = C^1B^1 : BC = SU : SS_1
= UV : S_1O^1 = UV : UW.

Hence VA : UV = WT : UW, and the angle between VA and WT or OH is equal to that between C^1B^1 and BC, which is the same as WUV.

Therefore, the triangles UVA, UWT and therefore also, UWV, UTA are similar. In other words, UOS and UTA are similar, since SWVO is cyclic.

13. *To prove that OH.OS = 2R.d.* (second method)

For VA : UA = OA ; OU, since V, U are inverse points.
OH : VA = SS_1 : SU
UA : AT = SU : SO (by § 12);

Compounding, OH : AT = OA : SO
∴ OS.OH = OA.AT = 2Rd, since AT = 2d, as in § 5.

14. *To prove that SS_1, HH_1 meet AO produced in inverse points X, Y.* For, $HH_1O = HCB - HBC$; and the angle between AO produced and OH_1 is equal to $C - B$.

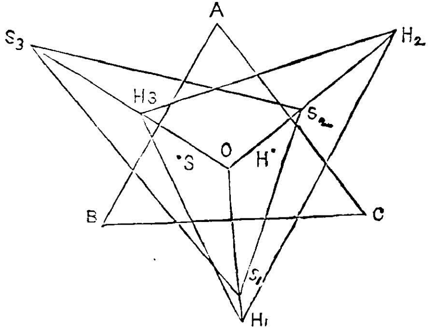
Therefore the angle between AO and $HH_1 = HCB - HBC - C + B = SBC - SCB = SS_1O$

Hence, the points X, Y, H_1, S_1 are cyclic, and $OX.OY = OS_1 . OH_1 = R^2$ (by § 1)

15. Let S_2, H_2 be the centres of the circles CSA, CHA; and S_3, H_3 of ASB, AHB. Also, let S'', S''', H'', H''' denote the reflections of S, H in the other two sides. *Then $S_1S_2S_3$ is similar to $H'H''H'''$ and $H_1H_2H_3$ to $S'S''S'''$.*

For, the sides of $S_1S_2S_3$ are $\perp^r$ to SA, SB, SC and therefore parallel to those of the pedal triangle of H, which is homothetic to $H'H''H'''$

Similarly for $H_1H_2H_3$ and $S'S''S'''$.



16. *O, H are corresponding points in the similar figures $S_1S_2S_3$ $H'H''H'''$; and similarly S, O in $S'S''S'''$, $H_1H_2H_3$.*

17. *The circumcentre O is the isogonal conjugate of S and H with respect to $S_1S_2S_3$ and $H_1H_2H_3$ respectively*

For, $S_1 = \pi - B_1SC = OS_1B_1$ since S_1 is the centre of BSC, and therefore $SS_1S_3 = BS_1S_3 = OS_1S_2$. Hence, etc.

Similarly, for H and O.

18. *The pedal circle of S or O with respect to $S_1S_2S_3$ is equal to that of H or O with respect to $H_1H_2H_3$.*

For, each is equal to a circle of radius $\frac{1}{2}$R.

19. *To prove that the following pairs of triangles are similar:*

S_3SS_2, BHC; S_1SS_3, CHA; S_2SS_1, AHB; H_3HH_2, BSC; H_1HH_3, CSA; H_2HH_1, ASB.

In the first pair

$$SS_3S_2 = \tfrac{1}{2}\, SS_3A = SBA = HBC;$$
$$SS_2S_3 = \tfrac{1}{2}\, SS_2A = SCA = HCB.$$

Hence, the triangles are equiangular, &c.

20. If S and H are the Brocard points of the triangle ABC, the following properties are easily deduced:—

(i) ABC, $S_2S_3S_1$, $H_3H_1H_2$ are similar,

(ii) $S_2S_3S_1$ is *congruent* to $H_3H_1H_2$,

(iii) S, O are the Brocard points of $S_2S_3S_1$, and O, H of $H_3H_1H_2$.

(iv) The circumradius of either circle $= \frac{1}{2}$R cosec ω

(v) The circumcentres lie on opposite sides of O at equal distances.

On Equations for determining $\cos\pi/(2n+1)$.

By K. J. Sanjana, M.A.

1. Taking the equation $\cos 2n\theta + \cos\theta = 0$, we have, $\cos 2n\theta = \cos(\pi-\theta)$, so that $2n\theta = (2r\pm1)\pi\pm\theta$. Thus θ has two sets of values $k\pi/(2n\pm1)$, where k is an odd integer. The series $\cos k\pi/(2n-1)$ has n distinct values, of which one is $\cos\pi$ or -1; the series $\cos k\pi/(2n+1)$ has $n+1$ values, of which also one is -1. Thus there are altogether $2n$ distinct values of θ including -1. Hence $\cos 2n\theta + \cos\theta$ has $2n$ factors, one of which is $1+\cos\theta$; $n-1$ of these belong to the first series and n to the second. The products of the factors in these two sets will each be rational, because (1) $\cos 2n\theta+\cos\theta$ is a rational function of $\cos\theta$ of degree $2n$ of which one factor is $1+\cos\theta$, and (2) the surds involved in $\cos k\pi/(2n-1)$ and $\cos k\pi/(2n+1)$ are of different kinds, so that if the products of factors in each set were irrational, the product of the two sets could not be rational. Putting c_q for $\cos q\theta$ and c_1 or c for $\cos\theta$, we infer

$$c_{2n}+c_1=(1+c)\ \mathrm{F}_{n-1}\ \mathrm{F}_n,$$

where the F's represent algebraical rational functions of $\cos\theta$ of degrees given by the suffixes.

2. Taking the next value $n+1$ of the coefficient of θ, we obtain for θ besides the value π the two sets of values $k\pi/(2n+1)$, $k\pi/(2n+3)$. The first set is precisely the same as the second set in the solution of the previous equation; and by reasoning exactly as before we can infer

$$c_{2n+2}+c_1=(1+c)\ \mathrm{F}_n\ \mathrm{F}_{n+1},$$

where the last factor is an algebraical rational function of $\cos\theta$ of degree $n+1$.

Similar expresssions hold for $c_{2n+4}+c_1$, $c_{2n+6}+c_1$,......, the first F in each being the same as the second in the immediately preceding expression.

3. Employing the formula

$$\cos 2n\,\theta=(-1)^n\left\{1-\frac{(2n)^2}{2\,!}\cos^2\theta+\frac{(2n)^2[(2n)^2-2^2]}{4\,!}\cos^4\theta-\ldots\right\},$$

we can easily obtain the following results in succession :—

$$c_2+c_1=-(1+c)(1-2c)\ ;$$
$$c_4+c_1=(1+c)(1-2c)(1+2c-4c^2)\ ;$$
$$c_6+c_1=-(1+c)(1+2c-4c^2)(1-4c-4c^2+8c^3)\ ;$$
$$c_8+c_1=(1+c)(1-4c-4c^2+8c^3)(1+4c-12c^2-8c^3+16c^4)\ ;$$
$$c_{10}+c_1=-(1+c)(1+4c-12c^2-8c^3+16c^4)(1-6c-12c^2+32c^3+16c^4-32c^5)\ ;$$

&c. &c.

Equating the last factors on the right hand sides to zero, we obviously get equations to determine the cosines of $\pi/3$; $\pi/5, 3\pi/5$; $\pi/7, 3\pi/7, 5\pi/7$; $\pi/9, 3\pi/9, 5\pi/9, 7\pi/9$; $\pi/11, 3\pi/11, 7\pi/11, 9\pi/11$; and so forth.

4. In the fourth equation given above, one of the roots is $\cos 3\pi/9$, *i.e.* $\cos \pi/3$, and belongs to the first equation: we can, therefore, suppress the factor $1-2c$, and obtain the equation $1+6c-8c^3=0$ to determine $\cos \pi/9$, $\cos 5\pi/9$, $\cos 7\pi/9$.

Again from the equality

$$c_{14}+c_1=-(1+c)(1+6c-24c^2-32c^3+80c^4+32c^5-64c^6)\times$$
$$(1-8c-24c^2+80c^3+80c^4-192c^5-64c^6+128c^7),$$

which may be obtained by carrying the process further, we infer that $\cos(\pi/13)$ (1, 3, 5, 7, 9, 11) are the roots of

$$1+6c-24c^2-32c^3+80c^4+32c^5-64c^6=0;$$

while the last factor equated to zero gives the roots $\cos(\pi/15)$ (1, 3, 5, 7, 9, 11, 13). In the latter case $\cos 5\pi/15=\frac{1}{2}$, and $\cos 3\pi/15$, $\cos 9\pi/15$ are the roots of $1+2c-4c^2=0$: thus the factors $1-2c$, $1+2c-4c^2$ can be suppressed, and we get the equation

$$1-8c-16c^2+8c^3+16c^4=0$$

as one having $\cos\pi/15$, $\cos 7\pi/15$, $\cos 11\pi/15$, $\cos 13\pi/15$ for roots. Generally from the factors of $c_{2r}+c_1$ we get two equations, of which the first has for principal root $\cos\pi/(2r-1)$ and the second $\cos\pi/(2r+1)$. When $2r+1$ is a prime, the degree of the latter equation is r; but when it is a composite number, the degree reduces to the number of fractions in $(1, 3, 5, \ldots\ldots 2r-1)\div(2r+1)$ which are not reducible to lower terms. For example, the degrees of the equations to determine $\cos(\pi/19, \pi/21, \pi/23, \pi/25, \pi/27)$ are respectively 9, 6, 11, 10, 9.

5. Changing θ to $\frac{1}{2}\pi-\theta$ and employing the formula

$$\cos 2n\theta=1-\frac{[2n]^2}{2!}\sin^2\theta+\frac{[2n]^2([2n]^2-2^2)}{4!}\sin^4\theta\ldots\ldots,$$

we obtain the following identities :—

$$c_2-s_1=(1+s)(1-2s),$$
$$c_4+s_1=(1+s)(1-2s)(1+2s-4s^2),$$
$$c_6-s_1=(1+s)(1+2s-4s^2)(1-4s-4s^2+8s^3),$$
$$\&c. \qquad\qquad \&c.$$

Two other series of identities may be obtained from the two sets given above by changing θ to $\theta+\pi$.

6. Lastly, putting $2c=x$, we get in their simplest forms the equations for determining $\cos \pi/(2r+1)$. The following are the results in the first eight cases :—

$$x-1=0\;;\; x^2-x-1=0\;;\; x^3-x^2-2x+1=0\;;\; x^3-3x-1=0\;;$$
$$x^5-x^4-4x^3+3x^2+3x-1=0\;;$$
$$x^6-x^5-5x^4+4x^3+6x^2-3x-1=0\;;\; x^4+x^3-4x^2-4x+1=0,$$
$$x^8-x^7-7x^6+6x^5+15x^4-10x^3-10x^2+4x+1=0.$$

7. Of course, the equation for $\cos \pi/(2n+1)$ is most readily obtained from the identity $\cos (n+1)\theta+\cos n\theta=0$ by expansion. But the preceding method is more useful in obtaining in succession the equations for the various particular values of n. And it brings out other useful identities. Thus taking $c_{2n-p+1}+c_p=0$, which is true when $\theta=\pi/(2n+1)$, we see by similar reasoning that $1+c$ is a factor of the left side, and so is the expression F_n of § 1. The third factor will be the factor of the highest degree in the expression for $c_{2n-2p}+c_1$.

For example, taking the case of $n=5$, we get from the identities in § 3, the following results :—

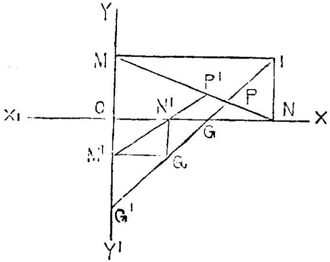
$$c_6+c_5=-(1+c)\ F_5,$$
$$c_7+c_4=(1+c)(1-2c)\ F_5,$$
$$c_8+c_3=(1+c)(1+2c-4c^2)\ F_5,$$
$$\text{and } c_9+c_2=-(1+c)(1-4c-4c^2+8c^3)\ F_5,$$

where $F_5\equiv1-6c-12c^2+32c^3+16c^4-32c^5$.

SHORT NOTES.

Normals to an ellipse from any given point.

Let a straight line MN of length $a+b$ slide between two rectangular lines XOX′, YOY′, Let it be divided at P into parts MP $=a$, PN $=b$. The locus of P, we know, is an ellipse. In fact, if the angle MNO $=\phi$, the co-ordinates of P are $a\cos\phi$, $b\sin\phi$ referred to OX, OY as axes, and P describes the ellipse $x^2/a^2+y^2/b^2=1$, which we shall call E.



The instantaneous centre for the motion of MN is the corner I of the rectangle OMIN. Hence IP is the normal at P to the ellipse E. Let Q be any point on this normal. Then P is the foot of one of the normals from Q to the ellipse E.

From Q draw QM′, QN′ perpendicular to the axes YY′, XX. Let P′ be the point where M′N′ (produced) cuts MN. Then it is a property of the figure that $\frac{M'P'}{P'N'}=\frac{a}{b}$. To show this, let the normal cut the axes in G, G′. From the triangle OM′N′ whose sides are cut by the transversal MN we have, (considering only the absolute values of the ratios involved),

$$\frac{M'P'}{P'N'}\cdot\frac{N'N}{NO}\cdot\frac{OM}{MM'}=1.$$

But $$\frac{N'N}{NO}=\frac{QI}{IG'},\quad \frac{OM}{MM'}=\frac{GI}{IQ}.$$

$$\therefore\quad \frac{N'N}{NO}\cdot\frac{OM}{MM'}=\frac{GI}{G'I}$$

$$\therefore\quad \frac{M'P'}{P'N'}=\frac{G'I}{GI}=\frac{G'M}{NI}=\frac{MP}{PN}=\frac{a}{b}.$$

From this property we have the following construction for the normals to the ellipse E from any given point Q. Draw QM′, QN′ perpendicular to the minor and major axes of E and take P′ in M′N produced such that $\frac{M'P'}{P'N'}=\frac{a}{b}$. *Through P′ draw a line cutting the minor and major axes in M, N so that the intercept MN may be equal to $a+b$.* In

MN take P such that $MP=a$, $PN=b$. Then P is the foot of one of the normals from Q. Drawing *all* the possible lines MN, we get the feet of all the normals from Q.

The problem of drawing normals to an ellipse is thus reduced to another having a classical interest.

The Elliptic Compass, we know, is one devised for the description of ellipses, based on the property stated at the beginning of this note. Making the line MN move between the guide we could practically note the positions for which MN, or MN produced, passes through any assigned point P′. Accordingly we see that the normals to an ellipse from any given point can be constructed by means of the Elliptic Compass by which the ellipse itself is described.

A given ellipse can be described in *two* ways by means of the elliptic compass; and it will now be obvious that we have two corresponding constructions for the feet of the normals.

V. RAMASWAMI AIYAR.

Graphic representation of the infinite series

$$\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}+\quad.\quad.\quad.\quad.\quad.$$

Many fundamental algebrical formulae are capable of being illustrated by diagrams. It is proposed to explain in this note how the sum of the infinite series $\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}+$, ... for a positive integral value of n may be geometrically determined.

1. In Fig. 1, ABCD is a rectangle, with the side AB divided into n equal parts. Let the several points of section be $P_1\, P_2 \ldots\ldots P_{n-1}$, so that AP_1 is the first part and $P_{n-1}B$ the last. Let this last part be similarly divided into n equal parts at $\theta_1\ \theta_2 \ldots\ldots \theta_{n-1}$; and so on. Erect perpendiculars at the points of sections.

Now, the first set of perpendiculars divides the whole rectangle into n equal parts, *i.e.* each part is $\frac{1}{n}^{th}$ of the whole; the second set divides one of the former into n parts again, so that each of these parts is $\frac{1}{n^2}$ of the whole; and so on.

Suppose that the *first* part of *each set* is coloured; then clearly the sum of the coloured portions is $\left(\frac{1}{n}+\frac{1}{n^2}+\ldots\ldots\right)^{th}$ of the whole.

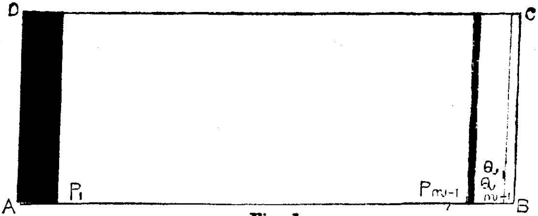


Fig. 1

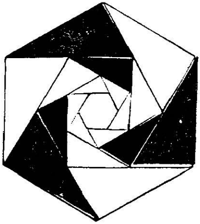


Fig. 3

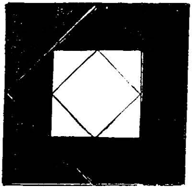


Fig. 2

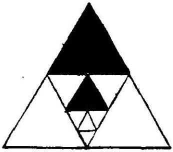


Fig. 4

Also, for *each* coloured strip there are $(n-2)$ *equal* strips left uncoloured, *i.e.*, the entire uncoloured portion is $(n-2)$ times the coloured; $\therefore$ the whole is $(n-1)$ times the coloured portion *i.e.*, the coloured portion is $\frac{1}{n-1}$ of the whole. Thus

$$\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}+\ldots\ldots\ldots\text{to infinity}=\frac{1}{n-1},$$

[*Note*. *A straight line* similarly treated will do for the purpose of demonstration. But, I believe, the effect on the eye is both pleasing and conclusive if we take a rectangle,]

2. Figures 2, 3 and 4 illustrate respectively the following particular cases :—

$$\text{(i)}\ \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\ldots\ldots=1,$$

$$\text{(ii)}\ \frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots\ldots\ldots=\tfrac{1}{2},$$

$$\text{(iii)}\ \frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\ldots\ldots\ldots=\tfrac{1}{3}.$$

In Fig. 2, each set of four triangles at the corners of a square covers in area half the square, and the space enclosed by them is the remaining half of the square. Thus the several sets of four corner triangles represent $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}$ .... of the whole square, and the construction being continued indefinitely, result (i) follows. This demonstration is taken from T. Sundara Rao's *Paper-Folding* (p. 4, 1893, Edn.), which also suggested the possibility of a similar demonstration in the other cases considered here. Figures 3 and 4 are similarly explained.

3. It may be pointed out that the method lends itself to the summation of other more complicated series and may be manipulated in a variety of ways.

N. Sanjiva Rao.

The foci of the general conic.

Let the conic be $ax^2+2hxy+by^2+2gx+2fy+c=0$, and ξ, η a focus. Transform the origin to ξ, η. The equation becomes

$$ax^2+2hxy+by^2+2g'x+2fy+c'=0$$

where $g'=a\xi+h\eta+g$; $f'=h\xi+b\eta+f$;

$$\text{and } c'=a\xi^2+2h\xi\eta+b\eta^2+2g\xi+2f\eta+c$$

Now *any* two perpendicular lines through the focus are conjugate. Thus $lx+my=0$ and $mx-ly=0$ are conjugate whatever $l:m$ may be; these lines are supposed to be referred to the new origin. Hence

$$\begin{vmatrix} a & h & g' & l \\ h & b & f' & m \\ g' & f' & c' & 0 \\ m & -l & 0 & 0 \end{vmatrix} = 0$$

must be true for all values of $l:m$. This equation when expanded becomes

$$(l^2+m^2)(g'f'-c'h)+lm\{c'(b-a)+g'^2-f'^2\}=0;$$

$$\therefore g'f'=c'h \text{ and } c'(b-a)+g'^2-f'^2=0;$$

$$\therefore \frac{g'f'}{h}=\frac{g'^2-f'^2}{a-b}=c'.$$

Hence the foci are given by the equations

$$\frac{(a\xi+h\eta+g)(h\xi+b\eta+f)}{h}=\frac{(a\xi+h\eta+g)^2-(h\xi+b\eta+f)^2}{a-b}$$

$$=a\xi^2+2h\xi\eta+b\eta^2+2g\xi+2f\eta+c$$

R. P. PARANJPYE.

On the face of the Sky for April and May.

Sidereal time at 8 p m.

		H.	M.	S.			H.	M.	S.
April 5	...	8	51	6	May 3	...	10	41	30
„ 12	...	9	18	42	„ 10	...	10	9	6
„ 19	...	9	46	18	„ 17	...	11	36	42
„ 26	...	10	13	54	„ 24	...	12	4	18
					„ 31	...	12	31	54

From this table the constellations visible during the evenings of April and May can be ascertained by a reference to their position as given in a star chart.

Phases of the Moon.

			H. M.			H. M.
First Quarter	... April	6 ...	11 . 25 A.M	May 5	...	6 . 44 P.M.
Full Moon	... „	13 ...	8 . 7 P.M.	„ 13	...	11 . 40 A.M.
Last Quarter	... „	22 ...	0 . 6 A.M.	„ 21	...	2 . 53 P M.
New Moon	... „	29 ...	3 . 55 A.M.	„ 28	...	11 . 54 A.M.

Planets.

Mercury—is an evening star attaining its greatest elongation E 19° 22′ on April 15. It is stationary on April 25, and reaches inferior

conjunction on May 5, after which date it is a morning star. It is again stationary on May 18. It is in conjunction with the **moon** on April 29 and on May 26, with **saturn** on May 10 and 29.

Venus is an evening star; It is in conjunction with the **moon** on April 1, on May 1 at 6^h 11^m and on May 31; with **Neptuue** on May 30.

Mars is in conjunction with the **moon** on April 24 at 6^h 41^m A M. and on May 19 at 7^h 33^m A.M.

Jupiter continues to retrograde ; and on May 1 he is in opposition to the sun. He is in conjunction with the **moon** on April 15 and on May 12.

Saturn is getting nearer the sun. On April 30 he is in conjunction with the **sun**. On April 29 and on May 27 he is in conjunction with the **moon**.

Uranns is stationary on May 5, after which date he retrogrades. He is in conjunction with the **moon** on April 21 and on May 19 at 2^h. 55^m A.M.

Neptune is in conjunction with the **moon** on April 6 and May 3 at 11^h P M. and on May 31.

Solar Eclipse.

On April 29, there is a total solar eclipse invisible in Europe, Asia, or Africa. The central line will pass from the south-east coast of Australia in a north-eastern direction over the Pacific Ocean nearly reaching Central America. It will be seen as a partial eclipse in the early morning in Tasmania and New Zealand, and in the late evening in the south-eastern parts of United states and in Cuba and its neighbourhood.

The totality will cross land only over some of the small islands in the central Pacific Ocean. The duration will be longest—5^m on the Christmas Island. This island derives its name from the fact that Captain Cook discovered it in his last voyage on the 24 December 1877. He celebrated Christmas there, and remained off the island until the end of the year. Cook called it Turtle Island from the abundance of turtle found there. It is about 60 miles in circuit with a large lagoon and has good anchorage on the western side. For many years it was much resorted to for guano.

Cook observed on the island, as a partial eclipse, the total solar eclipse of 29 December 1877. The latitude of the island is about 2° N, and longitude 157° W. (see : Mr. Lynn, in the Journal of the British Astronomical Association.)

V. RAMESAM.

SOLUTIONS.

Question 131.

(W. J. Greenstreet, M.A.) :—The line PQ cuts the sides BC, DA of a quadrilateral ABCD in P and Q respectively. AC, BD cut in S; AP, BQ cut in T; CQ, DP cut in U, and BC, AD in E. Shew that S,T,U are collinear, and that

$$S\{PCUD\} = \{EAQD\} = S\{PCTD\}.$$

Solution by S. Narayanan, B.A., L.T., V. B. Naik, M. A. and S. Purushottam, B.A.

The hexagon QBDPACQ is inscribed in the two line-conic AD, BC, applying Pascal's Theorem to the hexagon QBDPACQ, we immediately find that S, T, U are collinear.

Again $S(PCUD) = S(PCTD)$

$= (PMUD)$, where SC cuts DP in M

$= C(PMUD)$

$= (EAQD)$.

Analytical solution by N. Sankara Aiyar, B.A.

Question 175.

(M. Bhimasena Rao.) :—The circle through the points of contact of an inscribed conic of a triangle being called the *contact circle* of the centre of the conic, shew that the locus of a point whose contact circle passes through a given point P is a cubic passing through the angular points of the triangle and cutting the sides in three collinear points. Find the locus of P when the cubic breaks up into a line and a conic; and show that the contact circles of the intersections of the line and the conic in that case touch the locus of P.

Solution by S. Purushottam, B.A., S. Narayanan, B.A., L.T., and N. Sankara Aiyar, B.A.

Lemma: The contact circles of two isogonal conjugate points are coincident.

For, if X, Y be the conjugate points of ABC and PQR the contact circle of X, AX, BX, CX evidently bisect the sides of the $\triangle PQR$ Let the circle PQR cut the sides of ABC again in P′, Q′, R′; then the sides of P′Q′R′ are antiparallel to those of PQR, and therefore AY, BY, CY bisect the sides of P′Q′R′. In other words, P′Q′R′ is the contact circle of Y. Hence, the lemma is true.

1. Now, the equation of an in-conic whose centre is (f, g, h) is $\Sigma(\sqrt{a\alpha F}) = o$, where $F \equiv bg + ch - af$. (*vide*: Q. 215)

Therefore, the equation of the contact circle is

$$\Sigma(a\beta\gamma) = \Sigma(\lambda\alpha).\ \Sigma(a\alpha),$$

where $4bc\lambda = (bg + ch - af)\ (b'g + c'h - a'f)$, &c.

If this passes through a fixed point $(\alpha'\beta'\gamma')$, the locus of (f,g,h) is

$$4\,abc\Sigma(\alpha\beta'\gamma')=\Sigma(a\alpha').\ \Sigma\{\,a\alpha'(b\beta+c\gamma-a\alpha)(b/\beta+c/\gamma-a/\alpha)\,\}.$$

This is a cubic whose isogonal transformation is *itself* and is easily reduced to the form

$$\Sigma\{\,L(\beta/\gamma+\gamma/\beta)\,\}=K, \text{ where } L\equiv a,\ (a\alpha'-b\beta'-c\gamma')/a, \text{ \&c.}$$

2. Put $\alpha=0$, in the equation of the cubic and we get $M\gamma+N\beta=0$ which shows that the cubic circumsribes the triangle ABC and cuts its sides in points lying on the straight line $\alpha/L+\beta/M+\gamma/N=0$.

3. If the cubic breaks up into a line and a conic, the factors must be of the form

$$(l\alpha+m\beta+n\gamma)\,(l/\alpha+m/\beta+n/\gamma)=0$$

Comparing coefficients, we have

$$L=\lambda mn,\ M=\lambda nl,\ N=\lambda lm,\ K=-\lambda\ (l^2+m^2+n^2)\ ;$$

$$\therefore\ \sqrt{LMN}=\lambda^{3/2}\,lmn, \text{ \&c.}$$

Hence, the locus of P is $\sum\frac{\sqrt{MN}}{L}+K=0$.

4. Since, in the general case, the contact circles of points on the cubic pass through P, it is clear that when the cubic degenerates into a line and a conic, the contact circles of the double points of the degenerate cubic, (*viz*, the intersections of the line and the conic) must touch the locus of P found in § 3.

Question 179.

(V. Ramaswami Aiyar, M.A.):—Let O be the circumcentre, P the orthocentre and I the centre of a circle touching the sides of a triangle ABC. If the reflexions of AO, AP in IO, IP respectively cut at a, show that a lies in the side BC.

If AL, AM be the perpendiculars from A drawn to IO, IP and X be the middle point of A a, show that XL=XM. Also show that the circle whose centre is X and radius XL or XM touches the nine points circle of ABC at the corresponding Feuerbach's point.

Solution by S. Purushottam, B.A.

With A as a focus describe a conic to touch IP, IO at P and O; then a will be the other focus.

Now, as P and O are isogonal conjugate points, a confocal conic can be inscribed in the triangle ABC (*vide*: Vol. I, p. 92, J.I.M.C). Therefore a is the isogonal conjugate of A, and must therefore lie on the base BC.

Again, L and M are points on the auxiliary circle of the first conic. Therefore XL=XM. Also, by Mr. Ramaswami Aiyar's Theorem referred to on p. 98, Vol. I, the mutual orthoptic circle of the conic and the inscribed confocal touches the N.P.C. Hence the circle with centre X, and radius XL touches the N.P.C.

Lastly, suppose ABC is acute-angled; then the conic is a hyperbola and XN.XI is negative. Therefore X and I lie on the same side of N. And $XL=\frac{1}{2}(AO-Oa)$, since O is a point on the hyperbola; so that $XL<\frac{1}{2}R$. As the Feuerbach point is in NI produced, and as X, I are on the same side of N, it follows that the point of contact of the circle (X) and the N.P.C. is the Feuerbach point.

When A is obtuse, the conic in an ellipse, the radius of $(X)>\frac{1}{2}R$ and X and I lie on opposite sides of N; and a similar result follows.

Question 233.

(S. Narayanan, B A, L.T.) :—S and S′ are the foci of an ellipse whose eccentricity is greater than $1/\sqrt{2}$. P is a variable point on the circle on SS′ a diameter and PT, PT′ are tangents to the ellipse. Shew that the circumcentre of the triangle PTT′ lies on the major axis of the ellipse.

Remarks by N. Sankara Aiyar, B.A.

The centre of this circle is the middle point of the join of P and its *normal pole* (, *vide p.* 218, Casey's Anal. Geom., 2nd Edn.)

The normal pole of (x_1, y_1) has the coordinates

$$-c^2x_1\,(y_1^2-b^2)/(a^2y_1^2+b^2x_1^2)\ ;\ c^2y_1\,(x_1^2-a^2)/(a^2y_1^2+b^2x_1^2)$$

Hence the centre is

$$\tfrac{1}{2}\,x_1.\frac{b^2(x_1^2+y_1^2+c^2)}{a^2y_1^2+b^2x_1^2}\ ;\qquad \tfrac{1}{2}\,y_1\frac{a^2(x_1^2+y_1^2-c^2)}{a^2y_1^2+b^2x_1^2}.$$

If P be on the circle on SS′ as diameter, $x_1^2+y_1^2=c^2$;

that is, the y-coordinate of the required circum-centre is zero.

Hence this point lies on the major axis.

Question 234.

(Unity) :—Give an analytical proof for the following :—

Three tangents to a parabola at the points A, B, C form an equilateral triangle. Prove that the ellipse which touches the triangle ABC at the middle points of the sides passes through the focus of the parabola.

Solution by N. DuraiRajan.

The triangle PQR is equilateral and the focus O is on the circum circle of PQR.

$$\therefore\ OP \pm OQ \pm OR = 0.$$

Also, AB, BC, CA subtend angles of 120° each at the focus O

$$\therefore \frac{\triangle BOC}{OB.OC} = \frac{\triangle COA}{OC.OA} = \frac{\triangle AOB}{OA.OB}.$$

But $OB.OC = OP^2$, etc.

$$\therefore \frac{\triangle BOC}{OP^2} = \frac{\triangle COA}{OQ^2} = \frac{\triangle AOB}{OR^2}.$$

Now, the areal equation of the Steiner ellipse of the medial triangle is $\sqrt{x}+\sqrt{y}+\sqrt{z}=0$

The focus O lies on it, if $\sqrt{OP^2}+\sqrt{OQ^2}+\sqrt{OR^2}=0$, that is, if $OP \pm OQ \pm OR = 0$, which is true.

Question 238.

(M. Bhimasena Rao).—Find the locus of orthopoles, with respect to a triangle, of lines through a given point. Determine the position of the given point when the locus is (1) a circle and (2) an inconic of the triangle.

Solution by S. Purushottam, B.A., and 'Zero.'

1. The orthopole of a line through a given point P may be constructed as follows:—Describe semicircles on PB, PC to cut the given line in K, K'. Through K, K' draw perpendiculars to AC, AB; the intersection of these lines is the orthopole required (*vide*: J. I. M. C. vol. I, p. 137.)

Taking PB, PC, as the co-ordinate axes, the circles may be written in the form

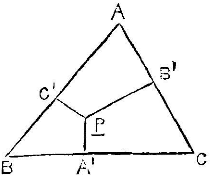
$$x^2+y^2-2\,xy\cos A = 2\,gx+2\,fy,$$

$$\text{and } x^2+y^2-2\,xy\cos A = 2\,g'x+2\,f'y.$$

If $y=mx$ be *any* line through P, the *orthopole* is found from the equations

$$xm^2-2\,m(x\cos A+f)+x-2\,g=0,$$

$$\text{and, } (y-2\,f')m^2-2m\,(y\cos A+g')+y=0.$$



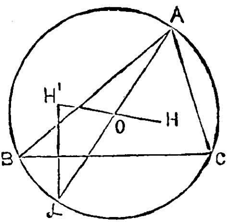
Hence, the locus of the orthopole is got by eliminating m between these equations and is readily seen to be a quadratic in (x, y). In other words, the locus is a conic.

2. The points of intersection of this conic with the sides of ABC are easily found.

Since A′, B′, C′ are the orthopoles of PA′, PB′, PC′ they are points on the conic.

To find the other points of intersection: let α be the point diametrically opposite to A on the circumcircle ABC and α′ the other extremity of the chord through α and P. Draw α′ A″ ⊥′ on BC. Then A″ is a point on the conic, since it is the orthopole of Pα, being the intersection of the Simson lines of α and α′.

3. If the conic is an in-conic, A′A′ vanishes and Pα coincides with PA′. In this case P must coincide with H′ on HO, such that H′O = OH. Also, AO, BO, CO being isogonally conjugate to AH, BH, CH bisect the sides of A′B′C′; so that O is the centre of the inconic.



4. The conic coincides with the N.P.C, when P is at O.

Question 241.

(R. Srinivasan, M.A.):—Show that

$$\sum_{n=o}^{n=a} \frac{1-n(n-1)x^2}{x^2+\{1+n(n-1)x^2\}^2} = \frac{a}{1+a^2x^2}.$$

Additional Solution by S. Narayanan, B.A., L.T.

Here, the general term is equal to

$$\frac{1-n(n-1)x^2}{1+(2n^2-2n+1)x^2+n^2(n-1)^2x^4}$$

$$=\frac{n}{1+n^2x^2}-\frac{n-1}{1+(n-1)^2x^2}.$$

Hence $S=\frac{a}{1+a^2x^2}.$

Question 246.

(T. Rajarama Rao, B.A., B.L.):—O is the middle point of a chord PQ of a conic. Two chords AOC and BOD are drawn making equal angles with the axes. Circles are described through A,O,D and B,O,C cutting PQ at E and F. Prove that A,B,C,D,E, and F lie on a conic whose axes are parallel to the axes of the given conic.

Solution by S. Purushottam, B.A. and A. Krishnaswami Iyengar.

Let PQ cut AD and CB in E′ and F′.

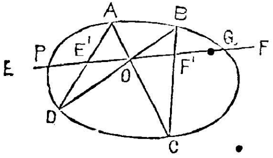
Describe the circle through ABCD cutting PQ at P′ and Q′. Then we know that PQ is parallel to the polar of O with respect to *any* conic through ABCD.

Hence, P′Q′ and E′F′ are bisected at O, being chords through O of the circle and the two-line conic through ABCD.

$$\text{Now } AE'.E'D = P'E'.E'Q' = Q'F'.F'P' = BF'.F'C$$

$$\therefore OE'.E'E = OF'.F'F. \quad \text{But } OE' = OF'. \quad \therefore OE = OF.$$

Since EF is bisected at C, the conic through A,B,C,D and E must also pass through F.



Again, because A,B,C,D, are concyclic, the axes of any conic through ABCD are parallel to the bisectors of the angles between AB and CD.

Hence the axes of the two conics are parallel.

Question 252.

(R. Srinivasan, M.A.):—If O be a point such that OA.OB.OC is proportional to the circum-radius of the triangle whose vertices are the feet of the perpendiculars from O on the sides of the triangle ABC, find the locus of O.

Solution (I) by M. Bhimasena Rao, (II) by S. Narayanan B.A., L.T., and others.

(I) The radius ρ of the pedal circle of O is given by

$2\rho = \dfrac{AO.BO.CO}{R^2 - OS^2}$, where S and R are the circumcentre and the circumradius of the triangle ABC : For proof, see Ex. 1, page 69. Lachlan's Geometry.

If ρ : OA.OB.OC is given, $R^2 - OS^2$ is constant; and O evidently lies on a circle concentric with the circumcircle of ABC. In this case, the area of the pedal triangle of O is also constant.

(II) Since the sides of the pedal triangle of O are respectively OA sinA, OB sinB, OCsinC, the circumradius of that triangle is equal to

$$\frac{OA.OB.OC. \sin A \sin B, \sin C}{4 \times \text{area of that triangle}}$$

$$\therefore \frac{4 \times \text{area of that triangle}}{\sin A \sin B \sin C} = \text{const.}$$

Hence the locus of O is

$$\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C = \lambda ;$$

which is a circle concentric with the circumcircle.

Question 254 (2).

(V. Ramaswami Aiyar, M.A.) :—If points P, Q within a triangle ABC be isogonal conjugates, show that

$$\frac{AP.AQ}{AB.AC} + \frac{BP.BQ}{BA.BC} + \frac{CP.CQ}{CA.CB} = 1.$$

Solution by M. Bhimasena Rao and S. Purushottam.

Let DEF be the pedal triangle of P. The sides of DEF are perpendicular to the lines joining Q to A, B, C.

The area of the triangle ABC is equal to the sum of the areas AFQE, BDQF and CEQD ;

$\therefore \Delta = \frac{1}{2}$ (AQ . EF + BQ . FD + CQ . DE)

Now EF = AP. sin A, FD = BP sin B, CQ = DE. sin C ;

$\therefore$ AQ · AP sin A + BQ . BP sin B + CQ . CP sin C = 2Δ

Dividing by 2Δ, we have

$$\frac{AP.AQ}{AB.AC} + \frac{BP.BQ}{BC.BA} + \frac{CP.CQ}{CA.CB} = 1.$$

Question 260.

(S. Ramanujam *communicated by* Mr. P. V. Seshu Aiyar) :—Show without using the Calculus that

$$\frac{3}{2} \log 2 = 1 + \frac{2}{4^3 - 4} + \frac{2}{8^3 - 8} + \frac{2}{12^3 - 12} + \ldots \textit{ad inf.}$$

Solution by K. J. Sanjana, M.A. and S. Narayan, B.A., L.T.

The general term, $\frac{2}{(4n)^3 - (4n)}$, of the series on the right is

$$\frac{1}{4n-1} + \frac{1}{4n+1} - \frac{1}{2n} ;$$

so that $\sum_{1}^{\infty} 2 \div \{ (4n)^3 - (4n) \}$

$$= (\tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{2}) + (\tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{4}) + (\tfrac{1}{11} + \tfrac{1}{13} - \tfrac{1}{6}) + \quad \ldots \quad , (\alpha),$$

$$= (\tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} - \tfrac{1}{4} + \tfrac{1}{6}) + (\tfrac{1}{7} - \tfrac{1}{8} + \tfrac{1}{9} - \tfrac{1}{10} - \tfrac{1}{8} + \tfrac{1}{10}) + \ldots$$

$$=\left(\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\tfrac{1}{7}-\tfrac{1}{8}+\tfrac{1}{9}-\tfrac{1}{10}\ldots\ldots\right)-\tfrac{1}{2}\left(\tfrac{1}{2}-\tfrac{1}{3}+\tfrac{1}{4}-\tfrac{1}{5}+\ldots\right)$$

$=\log 2-\frac{1}{2}-\frac{1}{2}(1-\log 2)=\frac{3}{2}\log 2-1$, which proves the identify.

N.B.—The terms in (α) cannot be combined as below

$$-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\tfrac{1}{7}\quad\ldots\quad\ldots\quad\ldots$$

because then the last terms of the brackets in (α) have to be taken at double the rate at which the first two terms are taken.

(*Vide*: Whittaker's *Modern Analysis*, p. 21).

Question 268.

268. (R. Srinivasan, M.A.) :—Show that the set of equations:

$$\frac{x^4}{a^{\frac{5}{4}}\,b^{\frac{3}{4}}}+\frac{y^4}{a^{\frac{3}{4}}\,b^{\frac{5}{4}}}=mx^2;\qquad \frac{x^{\frac{3}{2}}}{(az)^{\frac{1}{4}}}=x+y=\frac{y^{\frac{3}{2}}}{(bz)^{\frac{1}{4}}},$$

is possible for real values of x, y, z only if m is greater than 512.

Solution by K. J. Sanjana M.A., Mukand Lal, M.A., and S. Narayanan, B.A., L.T.

Let $\dfrac{x^{\frac{3}{2}}}{(az)^{\frac{1}{4}}}=x+y=\dfrac{y^{\frac{3}{2}}}{(bz)^{\frac{1}{4}}}\equiv\theta$ (suppose) ;

then $\qquad x=\theta^{\frac{2}{3}}(az)^{\frac{1}{6}},\ y=\theta^{\frac{2}{3}}(bz)^{\frac{1}{6}}$,

$$\therefore\ x+y\equiv\theta=\theta^{\frac{2}{3}}z^{\frac{1}{6}}(a^{\frac{1}{6}}+b^{\frac{1}{6}});$$

that is $\theta^{\frac{1}{3}}=z^{\frac{1}{6}}(a^{\frac{1}{6}}+b^{\frac{1}{6}})$.

$$\therefore\ x=z^{\frac{1}{2}}a^{\frac{1}{6}}(a^{\frac{1}{6}}+b^{\frac{1}{6}})^2,$$

and $y=z^{\frac{1}{2}}b^{\frac{1}{6}}(a^{\frac{1}{6}}+b^{\frac{1}{6}})^2$

Substituting in the first equation given, we have

$$(z^2a^{\frac{2}{3}}b^{\frac{1}{2}}+z^2a^{\frac{1}{2}}b^{\frac{2}{3}})(a^{\frac{1}{6}}+b^{\frac{1}{6}})^8=mz^2a^{\frac{5}{4}}b^{\frac{5}{4}},$$

$$\text{or } (a^{\frac{1}{6}}+b^{\frac{1}{6}})^9=ma^{\frac{3}{4}}b^{\frac{3}{4}};$$

$$\therefore\left\{\left(\frac{a}{b}\right)^{\frac{1}{12}}+\left(\frac{b}{a}\right)^{\frac{1}{12}}\right\}^9=m$$

But the sum of a real quantity and its reciprocal is always greater than 2 ;

$$\therefore\left(\frac{a}{b}\right)^{\frac{1}{12}}+\left(\frac{b}{a}\right)^{\frac{1}{12}}>2$$

Hence m is greater than 2^9 or 512.

QUESTIONS FOR SOLUTION.

273. (S. NARAYANAN, B.A., L.T.):

Given that $a+b+c=0$......A,

and $(2-ab-ac-bc)l+mabc+1=0$......B,

find three functions ϕ_1, ϕ_2, ϕ_3 of a, b, c so that the relation

$$\frac{\phi_1(a,b,c)}{a^2+\lambda a+\mu}+\frac{\phi_2(a,b,c))}{b^2+\lambda b+\mu}+\frac{\phi_3(a,b,c)}{c^2+\lambda c+\mu}=0,$$

may be independent of a,b,c in virtue of the relations A and B.

Also, interpret the question geometrically.

274. (M. BHIMASENA RAO):—The contact circles of the points H_1 and H_2 obtained in § 1, in the note on "circles related to a triangle," No. 5, Vol. II of the Journal, intersect the nine point circle of the triangle ABC at the Tarry point of its medial triangle.

275. (M. BHIMASENA RAO):—Show that the foci of the three parabolas each touching two of the three sides of a triangle and also the join of their mid points at the point where this join is intersected by the corresponding side of the orthocentric triangle, are collinear.

276. (P. V. SESHU AYAR, B.A., L.T.):—Solve in positive integers the equations $x+y=a(z+w)$; $bxy=zw$; where a and b are positive integers and x,y,z,w are the unknowns.

277. (*Communicated by* MR. H. B. MASTER.):—In how many ways can a rupee be exchanged for smaller coins in accordance with the following table? 3 pies = 1 pice, 2 half-pice = 1 pice 2 pice, = 1 half-anna, 2 half-annas = 1 anna, 2 annas = 1 two-anna piece, 2 two-anna pieces = 1 quarter-rupee, 2 quarter-rupees = 1 half-rupee, 2 half-rupees = 1 rupee.

278. (K. J. SANAJANA, M.A.):—When m is an integer prove that $m(m+1)(11m^3+44m^2+71m+54)$ is a multiple of 20. Also examine when the expression is a multiple of 80.

279. (K. J. SANJANA, M.A.):—A candidate appears at an examination in which there are p papers each carrying m marks. Prove that the greatest number of ways in which any marks could be scored is when the candidate gets 50 per cent of the total; it being given that no fractional mark is assigned, and that mp is an even integer.

280. (T. RAJARAMA RAO, B.A., B.L.):—ABC is a triangle of which O is the circum-centre. D and E are the intersections of BC with another circle which touches the circumcircle at A. F is so taken on AB as to make DF and CF equally inclined to AB. G is a simliar

point on AC. EF and DG intersect at P and the line AO meets BC at Q. Prove that

$$\frac{BD \cdot CQ}{CE \cdot BQ} = \frac{PD}{PE}.$$

281. (J. C. SWAMINARAYAN, M.A.) :—In a triangle ABC, straight lines AX, BY, and CZ are drawn, making angles α, β, γ with the sides AB, BC, CA respectively. Shew that the area of the triangle formed by AX, BY and CZ is

$$\frac{2R^2\{\sin(A-\alpha)\sin(B-\beta)\sin(C-\gamma)-\sin\alpha\sin\beta\sin\gamma\}^2}{\sin(B-\beta+\gamma)\sin(C-\gamma+\alpha)\sin(A-\alpha+\beta)},$$

where R is the circum-radius of $\triangle ABC$

282. (A. KRISHNASWAMI IYENGAR.) :—From the vertex of the Folium of Descartes, any line is drawn to cut it in two other points; shew that the intercept between these points subtends a right angle at the origin.

283. (S. RAMANUJAM) :—Shew that it is possible to solve the equations

$$x+y+z=a, \qquad px+qy+rz=b,$$
$$p^2x+q^2y+r^2z=c, \qquad p^3x+q^3y+r^3z=d,$$
$$p^4x+q^4y+r^4z=e, \qquad p^5x+q^5y+r^5z=f,$$

where x,y,z,p,q and r are the unknowns. Solve the above when $a=2$, $b=3$, $c=4$, $d=6$, $e=12$ and $f=32$.

284. (S. RAMANUJAM) :—Solve $\dfrac{x^5-6}{x^2-y}=\dfrac{y^5-9}{y^2-x}=5\,(xy-1)$

285. (N. DURAI RAJAN) .—In the *second* line of Q. 250 *read*: "the three diagonals are mutually at right angles and meet in a point O"; and show that the twenty-four-point sphere, the auxiliary sphere and the circum-sphere have a common radical plane and that the point O is a point-sphere of the coaxal system.

286. (UNITY) :—If on the perpendiculars at D, E, F, the mid-points of the sides of ABC, points P, Q, R are taken such that

$$DP/BC = EQ/CA = FR/AB,$$

prove that the triangles PQR, ABC are in perspective and that the axis of perspective envelopes a parabola.

287. (A. C. L. Wilkinson, M.A., F.R.A.S.) :—A point moves in a straight line with a velocity which is initially u, and which receives constant increments equal to u, at equal intervals T. Prove that the velocity at any time t after the beginning of the motion is

$$\frac{u}{2}+\frac{ut}{T}+\frac{u}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin\frac{2m\pi t}{T},$$

and that the distance traversed is

$$\frac{ut}{2T}(t+T)+\frac{uT}{12}-\frac{uT}{2\pi^3}\sum_{m=1}^{\infty}\frac{1}{m^3}\cos\frac{2m\pi t}{T}.$$

(*Whittaker's* Modern Analysis : p. 159, Ex. 1C)

288. (D. D. Kapadia, M.A., B.Sc.) :—Solve the equation

$$\begin{vmatrix} a^2+x, & ab, & ac, & ad, & \ldots\ldots al \\ ba, & b^2+x, & bc, & bd, & \ldots\ldots bl \\ ca, & cb, & c^2+x, & cd, & \ldots\ldots cl \\ \ldots & \ldots & \ldots & \ldots & \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots\ldots \\ la, & lb, & lc, & ld, & \ldots\ldots l^2+x \end{vmatrix} = 0.$$

289. (S. Ramanujam) .—Find the value of

(i) $1+\sqrt{\{1+2\sqrt{[1+3\sqrt{(1+\&c\ldots\ldots)}]}\}}$

(ii) $1+\sqrt{\{6+2\sqrt{[7+3\sqrt{(8+\&c\ldots\ldots)}]}\}}$